HARMONIC AND BIHARMONIC RIEMANNAIN SUBMERSIONS FROM SOL SPACE

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Abstract

In this paper, we give a complete classification of harmonic and biharmonic Riemannian submersions $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space into a surface by proving that there is neither harmonic nor biharmonic Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space no matter what the base space (N^2, h) is. We also prove that a Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space exists only when the base space is a hyperbolic space form.

1. Introduction and Preliminaries

All manifolds, maps, tensor fields studied in this paper are assumed to be smooth unless there is an otherwise statement.

Recall that a *harmonic map* $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the energy functional

$$E(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\mathrm{d}\varphi|^2 \,\mathrm{d}x.$$

The Euler-Lagrange equation is given by the vanishing of the tension filed $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ (see [7]). Clearly, the map φ is harmonic if and only if $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi = 0$ holds identically.

The study of biharmonic maps as a special case of k-polyharmonic maps were first proposed by J. Eells and L. Lemaire in [7]. A *biharmonic map* $\varphi : (M, g) \to$

Date: 10/06/2022.

1991 *Mathematics Subject Classification.* 58E20, 53C12.

Key words and phrases. Harmonic map, Biharmonic maps, Riemannain submersions, Sol space.

*Supported by the Natural Science Foundation of China (No. 11861022).

** Supported by a grant from the Simons Foundation (427231, Ye-Lin Ou).

(N, h) between Riemannian manifolds is a critical point of the bienergy

$$E^2(\varphi, \Omega) = \frac{1}{2}\int_\Omega |\tau(\varphi)|^2 \,\mathrm{d}x$$

for every compact subset Ω of M, where $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ is the tension field of φ. Jiang [9] first computed the first variation of the functional to see that φ is biharmonic if and only if its bitension field vanishes identically, i.e.,

$$\tau^2(\varphi) := \mathrm{Trace}_g(\nabla^\varphi\nabla^\varphi - \nabla^\varphi_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0, \tag{1}$$

where R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

We call a submanifold that is a biharmonic submanifold if the isometric immersion that defines the submanifold is a biharmonic map. Analogously, a Riemannian submersion is called a **biharmonic (respectively,harmonic) Riemannian submersion** if the Riemannian submersion is a biharmonic (respectively, harmonic) map. Obviously, any harmonic map is always biharmonic whilst biharmonic maps include harmonic maps as special cases. We use proper biharmonic maps (respectively, submanifolds, Riemannian submersion) to call those biharmonic maps that are not harmonic maps.

In the first part of the paper, we will study harmonicity and bharmoncity of Riemannian submersions from 3-dimensional Sol space into a surface. For harmonicity of Riemannian submersions, one of our motivations is that the definition of Riemannian submersions, in a sense, are considered as the dual notion of isometric immersions (i.e., submanifolds). There are many interesting examples of harmonic isometric immersions of a surface (i.e., minimal surfaces) into 3-manifolds, such as planes or catenoid in $\mathbb{R}^3$ or harmonic embedding of S^2 into S^3 [16]. On the other hand, there exist many interesting examples and classification results of harmonic Riemannian submersions from 3-dimensional Riemannian manifolds into a surface: Hopf fibration $\pi : S^3 \to S^2(4)$ and the orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ are harmonic Riemannian submersion; there is no harmonic Riemannian submersion $\pi : H^3 \to (N^2, h)$ no matter what (N^2, h) is (see [20, 24]); harmonic Riemannian submersions from Thurston's 3-dimensional geometries, 3-dimensional BCV spaces and a Berger sphere S^3_ε have been completely classified and many explicit constructions of harmonic Riemannian submersions were given (see [24] for details).

Since biharmonic maps are considered as the generalizations of harmonic maps and include harmonic maps as a subset, it would be very interesting to study

biharmonicity of Riemannian submersions. Based on this, we will study biharmonicity of Riemannian submersions from 3-dimensional Sol space into a surface in the second part of the paper. Biharmonic Riemannian submersions were first studied by Oniciuc in [13]. In [20], the authors first introduced so-called integrability data and then used the main tool to obtain a complete classification of biharmonic Riemannian submersions from a 3-dimensional space form into a surface. In [1], the authors studied biharmonicity of a general Riemannian submersion and obtained biharmonic equations for Riemannian submersions with one-dimensional fibers and Riemannian submersions with basic mean curvature vector fields of fibers, and they first used the so-called integrability data to study biharmonic Riemannian submersions from $(n+1)$-dimensional spaces with one-dimensional fibers. In [18], the author studied biharmonicity a more general setting of Riemannian submersion with a S^1 fiber over a compact Riemannian manifold. In [8] , the authors studied generalized harmonic morphisms and obtained many examples of biharmonic Riemannian submersions which are maps between Riemannian manifolds that pull back local harmonic functions to local biharmonic functions.

In addition to these, we refer the readers to the following classification results. In 2023, the authors [21] classified all proper biharmonic Riemannian submersions from BCV 3-diemnsional spaces into a surface. In a recent paper [22], the authors also gave complete classifications of biharmonic Riemannian submersions from 3-dimensional Berger sphere. And also, biharmonic Riemannian submersions from product spaces $M^2 \times \mathbb{R}$ to a surface have been completely classified in [23].

Recall that Sol space is one of Thurston's eight models of 3-dimensional geometry. It is the Riemannian manifold $(\mathbb{R}^3, g_{Sol})$, where the metric can be described by $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$ with respect to Euclidean coordinates on $\mathbb{R}^3$.

First of all, one observes that it is easy to find Riemannian submersions from Sol space. For example, the projections
$P_1 : (\mathbb{R}^3, g_{Sol}) \to (\mathbb{R}^2, e^{2z}\mathrm{d}x^2 + \mathrm{d}z^2),\ P_1(x, y, z) = (x, z)$, and
$P_2 : (\mathbb{R}^3, g_{Sol}) \to (\mathbb{R}^2, e^{2z}\mathrm{d}y^2 + \mathrm{d}z^2),\ P_2(x, y, z) = (y, z)$ are both Riemannian submersions.

One may wonder whether these are harmonic or biharmonic, whether there is any harmonic or biharmonic Riemannian submersions from Sol space. In this

paper, we prove that there is neither harmonic nor biharmonic Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space no matter what the base space (N^2, h) is. We also prove that a Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space exists only when the base space is a hyperbolic space form.

2. Harmonic Riemannian submersions from Sol space

In this section, we obtain a nonexistence classification results for harmonic Riemannian submersions from Sol space to a surface.
Let $(\mathbb{R}^3, g_{Sol})$ denote Sol space, where the metric $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$ with respect to local coordinates (x, y, z) in $\mathbb{R}^3$. We have a defined orthonormal basis as

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}.$$

With respect to this orthonormal frame, the Lie brackets and the Levi-Civita connection are given by:

$$[E_1, E_2] = 0,\ [E_2, E_3] = -E_2,\ [E_1, E_3] = E_1, \tag{2}$$

$$\begin{aligned} &\nabla_{E_1}E_1 = -E_3,\ \nabla_{E_1}E_3 = E_1,\ \nabla_{E_2}E_2 = E_3,\ \nabla_{E_2}E_3 = -E_2, \\ &\text{all other } \nabla_{E_i}E_j = 0,\ i, j = 1, 2, 3. \end{aligned} \tag{3}$$

One adopts the following notation and sign convention for Riemannian curvature operator.

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{4}$$

the Riemannian and the Ricci curvature tensors are given by

$$\begin{aligned} &R(X, Y, Z, W) = g(R(Z, W)Y, X), \\ &\mathrm{Ric}(X, Y) = \mathrm{Trace}_g R = \sum_{i=1}^{3} R(Y, e_i, X, e_i) = \sum_{i=1}^{3}\langle R(X, e_i)e_i, Y\rangle. \end{aligned} \tag{5}$$

A straightforward computation gives

$$\begin{aligned} &R_{1212} = g(R(E_1, E_2)E_2, E_1) = 1,\ R_{1313} = g(R(E_1, E_3)E_3, E_1) = -1, \\ &R_{2323} = g(R(E_2, E_3)E_3, E_2) = -1,\quad \text{all other } R_{ijkl} = 0, i, j, k, l = 1, 2, 3. \end{aligned} \tag{6}$$

Let $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ be a Riemannian submersion from Sol space with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$ on $(\mathbb{R}^3, g_{Sol})$ and e_3 being vertical. By a treatment similar to to those used treating Remark 1 in [21], we then have the following (7)–(13) (see [21] for details)

$$[e_1, e_3] = f_3 e_2 + \kappa_1 e_3,\ [e_2, e_3] = -f_3 e_1 + \kappa_2 e_3,\ [e_1, e_2] = f_1 e_1 + f_2 e_2 - 2\sigma e_3. \tag{7}$$

where $f_1, f_2, f_3,\ \kappa_1,\ \kappa_2$ and σ are the (generalizd) integrability data of the Riemannian submersion π. When $f_3 = 0$, the horizontal distribution $\{e_1,\ e_2\}$ are basic and $\{f_1, f_2, \kappa_1,\ \kappa_2,\ \sigma\}$ is the integrability data of the adapted frame.
The Levi-Civita connection is given by

$$\begin{aligned}
&\nabla_{e_1} e_1 = -f_1 e_2,\ \ \nabla_{e_1} e_2 = f_1 e_1 - \sigma e_3,\ \ \nabla_{e_1} e_3 = \sigma e_2,\\
&\nabla_{e_2} e_1 = -f_2 e_2 + \sigma e_3,\ \ \nabla_{e_2} e_2 = f_2 e_1,\ \ \nabla_{e_2} e_3 = -\sigma e_1,\\
&\nabla_{e_3} e_1 = -\kappa_1 e_3 + (\sigma - f_3) e_2, \nabla_{e_3} e_2 = -(\sigma - f_3) e_1 - \kappa_2 e_3, \nabla_{e_3} e_3 = \kappa_1 e_1 + \kappa_2 e_2.
\end{aligned} \tag{8}$$

Denoting by $e_i = \sum_{j=1}^{3} a_i^j E_j,\ i = 1, 2, 3$, using (3), (6) and (8), then the Jacobi identity applied to the frame $\{e_1, e_2, e_3\}$ gives

$$\begin{cases}
e_3(f_1) + (\kappa_1 + f_2) f_3 - e_1(f_3) = 0,\\
e_3(f_2) + (\kappa_2 - f_1) f_3 - e_2(f_3) = 0,\\
2e_3(\sigma) + \kappa_1 f_1 + \kappa_2 f_2 + e_2(\kappa_1) - e_1(\kappa_2) = 0,
\end{cases} \tag{9}$$

and the terms of the curvature tension as follows

$$\begin{cases}
R^M(e_1, e_3, e_1, e_2) = -e_1(\sigma) + 2\kappa_1\sigma = -2a_2^3 a_3^3,\\
R^M(e_1, e_3, e_1, e_3) = e_1(\kappa_1) + \sigma^2 - \kappa_1^2 + \kappa_2 f_1 = 2(a_2^3)^2 - 1,\\
R^M(e_1, e_3, e_2, e_3) = e_1(\kappa_2) - e_3(\sigma) - \kappa_1 f_1 - \kappa_1\kappa_2 = -2a_1^3 a_2^3,\\
R^M(e_1, e_2, e_1, e_2) = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma - 3\sigma^2 = 2(a_3^3)^2 - 1,\\
R^M(e_1, e_2, e_2, e_3) = -e_2(\sigma) + 2\kappa_2\sigma = 2a_1^3 a_3^3,\\
R^M(e_2, e_3, e_1, e_3) = e_2(\kappa_1) + e_3(\sigma) + \kappa_2 f_2 - \kappa_1\kappa_2 = -2a_1^3 a_2^3,\\
R^M(e_2, e_3, e_2, e_3) = \sigma^2 + e_2(\kappa_2) - \kappa_1 f_2 - \kappa_2^2 = 2(a_1^3)^2 - 1.
\end{cases} \tag{10}$$

Gauss curvature of the base space as

$$K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma, \tag{11}$$

$$e_3(K^N) = e_3[e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma] = 0. \tag{12}$$

When $f_3 = 0$, then Gauss curvature of the base space becomes

$$K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2. \tag{13}$$

Now we are ready to give the following classification of harmonic Riemannian submersions from Sol space.

Proposition 2.1. *(see[21]) A Riemannian submersion $\pi : (M^3, g) \to (N^2, h)$ is harmonic if and only if $\nabla^M_{e_3} e_3 = 0$, i.e., $\kappa_1 = \kappa_2 = 0$.*

Theorem 2.2. *There exists no harmonic Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space no matter what (N^2, h) is.*

Proof. Let $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ be a Riemannian submersion with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$, e_3 being vertical, and the (generalized) integrability data $\{f_1, f_2, f_3, \kappa_1, \kappa_2, \sigma\}$. By Proposition 2.1, the Riemannian submersion π is harmonic if and only if $\kappa_1 = \kappa_2 = 0$. Using (10) and Proposition 2.2 in [24], we obtain

$$\begin{cases} -e_1(\sigma) = -2a_2^3 a_3^3, \\ \sigma^2 = 2(a_2^3)^2 - 1, \\ -e_3(\sigma) = -2a_1^3 a_2^3 = 0, \\ K^N = 3\sigma^2 + 2(a_3^3)^2 - 1, \\ -e_2(\sigma) = 2a_1^3 a_3^3, \\ e_3(\sigma) = -2a_1^3 a_2^3 = 0, \\ \sigma^2 = 2(a_1^3)^2 - 1. \end{cases} \tag{14}$$

where $K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma$.
Comparing the 2nd equation with the 7th equation of (14), we have $(a_1^3)^2 = (a_2^3)^2$. However, using the 3rd equation of (14), we get $a_1^3 a_2^3 = 0$ and hence $a_1^3 = a_2^3 = 0$. We substitute this into the 2nd equation of (14) to have $\sigma^2 = -1$, a contradiction. From which we obtain the theorem.

□

3. Biharmonic Riemannian submersions from Sol space

We state the following proposition ([20]) which will be later used in the rest of the paper.

Proposition 3.1. *(see [20]) Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannaian submersion with the adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $f_1, f_2, \kappa_1,\ \kappa_2$* and σ.

Then, the Riemannaian submersion π is biharmonic if and only if

(15)
$$\begin{cases}-\Delta^M\kappa_1-2\sum\limits_{i=1}^{2}f_ie_i(\kappa_2)-\kappa_2\sum\limits_{i=1}^{2}(e_i(f_i)-\kappa_if_i)+\kappa_1\left(-K^N+\sum\limits_{i=1}^{2}f_i^2\right)=0,\\ -\Delta^M\kappa_2+2\sum\limits_{i=1}^{2}f_ie_i(\kappa_1)+\kappa_1\sum\limits_{i=1}^{2}(e_i(f_i)-\kappa_if_i)+\kappa_2\left(-K^N+\sum\limits_{i=1}^{2}f_i^2\right)=0\end{cases}$$

where $K^N=R^N_{1212}\circ\pi=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2$ is Gauss curvature of Riemannian manifold (N^2,h).

The following proposition was found in [21].

Proposition 3.2. *(see [21]) Let $\pi:(M^3,g)\longrightarrow(N^2,h)$ be a Riemannian submersion from 3-manifolds with an orthonormal frame $\{e_1,e_2,e_3\}$ and e_3 being vertical. If $\nabla_{e_1}e_1=0$, then either $\nabla_{e_2}e_2=0$, or $\nabla_{e_2}e_2\not\equiv0$, and the frame $\{e_1,e_2,e_3\}$ is an adapted frame.*

We will prove the important conclusion used proving our main theorem

Theorem 3.3. *Let $\pi:(\mathbb{R}^3,g_{Sol}=e^{2z}\mathrm{d}x^2+e^{-2z}\mathrm{d}y^2+\mathrm{d}z^2)\to(N^2,h)$ be a Riemannian submersion. Then, we have such an adapted frame $\{e_1=a_1^2E_2+a_1^3E_3,\ e_2,\ e_3\}$ of the Riemannian submersion π with e_3 being vertical. Moreover, if E_1 is not vertical, then $\nabla_{e_2}e_2\neq0$, i.e., $f_2\neq0$.*

Proof. Obviously, if E_1 is tangent to the fiber of the Riemannian submersion π , then any basic field is of the form $e=a^2E_2+b^2E_3$, and $a^2+b^2=1$.
From this time on, we only need to suppose that E_1 is not vertical, i.e., $e_3\neq\pm E_1$. Then, the vector filed $e_1=e_3\times E_1$ is horizontal and hence $\langle e_1,E_1\rangle=0$. From this, we have a defined orthonormal frame $\{e_1,\ e_2=e_3\times e_1,\ e_3\}$ on M^3. If denoting by $e_i=\sum\limits_{j=1}^{3}a_i^jE_j, i=1,2,3,$ together with $\langle e_1,E_1\rangle=0$, then e_1 is expressed as the form $e_1=a_1^2E_2+a_1^3E_3$ and hence $(a_1^2)^2+(a_1^3)^2=1$. From these, we have the following

$$a_1^1=0,\ a_3^1\neq\pm1\text{ and }a_2^1\neq0. \tag{16}$$

One can further check the following equalities as

$$f_1=0,\ \nabla_{e_1}e_1=0. \tag{17}$$

By a direct computation, we get

$$\nabla_{e_1}e_1=\nabla_{e_1}(\sum_{i=1}^{3}a_1^iE_i)=\sum_{i=1}^{3}e_1(a_1^i)E_i+\sum_{i,j=1}^{3}a_1^ja_1^i\nabla_{E_j}E_i. \tag{18}$$

However, using (8), the above has another expression as

$$\nabla_{e_1} e_1 = -f_1 e_2 = -f_1 \sum_{i=1}^{3} a_2^i E_i. \tag{19}$$

By equating (18) and (19) and comparing the coefficient of E_1, we obatin

$$\begin{aligned} -f_1 a_2^1 &= \langle -f_1 \sum_{i=1}^{3} a_1^i E_i, E_1 \rangle = \langle \nabla_{e_1} e_1, E_1 \rangle \\ &= \langle \sum_{i=1}^{3} e_1(a_1^i) E_i + \sum_{i,j=1}^{3} a_1^j a_1^i \nabla_{E_j} E_i, E_1 \rangle = e_1(a_1^1) = 0, \end{aligned} \tag{20}$$

which has been used (3) and $a_1^1 = 0$. This leads to $f_1 = 0$ for $a_2^1 \neq 0$, and hence (17) holds.
Applying (3), (8) and $a_1^1 = f_1 = 0$ and a further computation similar to those used calculating (18)–(20) gives

$$\begin{cases} e_1(a_1^2) = a_1^2 a_1^3, \\ e_1(a_1^3) = -(a_1^2)^2, \\ e_1(a_2^1) = -\sigma a_3^1, \\ e_1(a_2^2) = a_1^2 a_2^3 - \sigma a_3^2, \\ e_1(a_2^3) = -a_1^2 a_2^2 - \sigma a_3^3, \\ e_1(a_3^1) = \sigma a_2^1, \\ e_1(a_3^2) = a_1^2 a_3^3 + \sigma a_2^2, \\ e_1(a_3^3) = -a_1^2 a_3^2 + \sigma a_2^3, \\ f_2 a_2^1 = -a_1^3 a_2^1 + \sigma a_3^1 = -a_3^2 + \sigma a_3^1, \\ e_2(a_3^1) = -a_2^1 a_3^3, \\ e_2(a_2^1) = -a_2^1 a_2^3, \\ \kappa_1 a_3^1 = (\sigma - f_3) a_2^1 - a_1^3 a_3^1 = (\sigma - f_3) a_2^1 + a_2^2. \end{cases} \tag{21}$$

Since $\nabla_{e_1} e_1 = 0$, we conclude from Proposition 3.2 to have either $\nabla_{e_2} e_2 \neq 0$, and the frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π; or $\nabla_{e_2} e_2 = 0$. Now, we just need to consider the latter case $\nabla_{e_2} e_2 = 0$, i.e., $f_2 = 0$. From these, one has the following

$$a_1^1 = f_1 = f_2 = 0. \tag{22}$$

Then, (10) turns into

$$\begin{cases} -e_1(\sigma)+2\kappa_1\sigma=-2a_2^3a_3^3,\\ e_1(\kappa_1)+\sigma^2-\kappa_1^2=2(a_2^3)^2-1,\\ e_1(\kappa_2)-e_3(\sigma)-\kappa_1\kappa_2=-2a_1^3a_2^3,\\ 2f_3\sigma-3\sigma^2=2(a_3^3)^2-1,\\ -e_2(\sigma)+2\kappa_2\sigma=2a_1^3a_3^3,\\ e_2(\kappa_1)+e_3(\sigma)-\kappa_1\kappa_2=-2a_1^3a_2^3,\\ \sigma^2+e_2(\kappa_2)-\kappa_2^2=(2a_1^3)^2-1. \end{cases} \tag{23}$$

We now show that the latter case (i.e., $a_1^1=f_1=f_2=0$, $a_3^1\neq\pm1$ and $a_2^1\neq 0$) can not happen by considering the following two cases:

Case I: $a_3^1=0,\ f_2=0$. In this case, since $a_1^1=0$, we have $a_2^1=\pm1$ and hence $a_2^3=0$. By the 9th equation of (21), one easily sees that $a_3^2=a_3^1=0$ and hence $a_3^3=\pm1$. This leads to $a_1^3=0$ and $a_1^2=\pm1$. Substituting this into the 6th equation of (21), we have $\sigma=0$. However, we substitute $\sigma=0$ and $a_3^3=\pm1$ into the 4th equation of (23) to find $0=1$, a contradiction.

Case II: $a_3^1\neq 0,\ \pm1$ and $f_2=0$. In this case, since $a_1^1=0$, we then have $a_2^1\neq 0,\pm1$. Substitute $f_2=0$ into the 9th equation of (21) to have

$$a_3^2=\sigma a_3^1. \tag{24}$$

Applying e_1 to both sides the 12th equation of (21), we get

$$e_1(\kappa_1)a_3^1+\kappa_1e_1(a_3^1)=e_1(\sigma)a_2^1+\sigma e_1(a_2^1)-e_1(f_3)a_2^1-f_3e_1(a_2^1)+e_1(a_2^2), \tag{25}$$

which can be rewritten as

$$e_1(\kappa_1)a_3^1+\kappa_1e_1(a_3^1)-e_1(\sigma)a_2^1-\sigma e_1(a_2^1)+e_1(f_3)a_2^1+f_3e_1(a_2^1)-e_1(a_2^2)=0. \tag{26}$$

Using the 3rd, the 4th, the 6th equation of (21), the 1st, the 2nd equation of (23) and the 1st equation of (9), a straightforward computation gives

$$\begin{aligned} &0=(\kappa_1^2-\sigma^2+2(a_2^3)^2-1)a_3^1+\kappa_1\sigma a_2^1\\ &-(2\kappa_1\sigma+2a_2^3a_3^3)a_2^1+\sigma^2a_3^1+\kappa_1f_3a_2^1-f_3\sigma a_3^1-a_1^2a_2^3+\sigma a_3^2\\ &=\kappa_1^2a_3^1-\sigma^2a_3^1+2a_3^1(a_2^3)^2-a_3^1+\kappa_1\sigma a_2^1\\ &-2\kappa_1\sigma a_2^1-2a_2^3a_3^3a_2^1+\sigma^2a_3^1+\kappa_1f_3a_2^1-f_3\sigma a_3^1-a_1^2a_2^3+\sigma a_3^2. \end{aligned} \tag{27}$$

One substitutes the 12th equation of (21) and (24) into (26), together with $a_2^1a_2^3+a_3^1a_3^3=0$ and $a_2^2=a_1^3a_3^1$, to compute the following

$$\begin{aligned}
0 &= \kappa_1(\sigma a_2^1 - f_3a_2^1 + a_2^2) - \sigma^2 a_3^1 + 2a_3^1(a_2^3)^2 - a_3^1 + \kappa_1\sigma a_2^1\\
&\quad -2\kappa_1\sigma a_2^1 + 2a_3^1(a_3^3)^2 + \sigma^2 a_3^1 + \kappa_1 f_3 a_2^1 - f_3\sigma a_3^1 - a_1^2a_2^3 + \sigma^2 a_3^1\\
&= \kappa_1 a_2^2 + 2a_3^1[(a_2^3)^2 + (a_3^3)^2] - a_3^1 - f_3\sigma a_3^1 - a_1^2a_2^3 + \sigma^2 a_3^1\\
&= \kappa_1 a_3^1a_1^3 + 2a_3^1[1-(a_1^3)^2] - a_3^1 - f_3\sigma a_3^1 - a_1^2a_2^3 + \sigma^2 a_3^1\\
&= (\sigma a_2^1 - f_3a_2^1 + a_2^2)a_1^3 + 2a_3^1 - 2a_3^1(a_1^3)^2 - a_3^1 - f_3\sigma a_3^1 - a_1^2a_2^3 + \sigma^2 a_3^1\\
&= \sigma a_2^1a_1^3 - f_3a_2^1a_1^3 + a_2^2a_1^3 - 2a_3^1(a_1^3)^2 + a_3^1 - f_3\sigma a_3^1 - a_1^2a_2^3 + \sigma^2 a_3^1\\
&= 2\sigma^2 a_3^1 - 2f_3\sigma a_3^1 - 2a_3^1(a_1^3)^2,
\end{aligned} \tag{28}$$

the last equality holds by using the fact $a_2^1a_1^3=a_3^2$, $a_3^2=\sigma a_3^1$ and $a_3^1=a_1^2a_2^3-a_1^3a_2^2$. Since $a_3^1\neq 0$, then (28) becomes

$$2\sigma^2 - 2f_3\sigma - 2(a_1^3)^2 = 0 \tag{29}$$

Substituting the 4th equation of (23) into (29), together with $(a_1^3)^2+(a_2^3)^2+(a_3^3)^2=1$, and simplifying the resulting equation, we get

$$\sigma^2 = 2(a_2^3)^2 - 1. \tag{30}$$

This implies

$$\sigma^2(a_3^1)^2 = 2(a_2^3a_3^1)^2 - (a_3^1)^2. \tag{31}$$

Since $\sigma a_3^1 = a_3^2$, the above equation is equivalent to

$$2(a_2^3a_3^1)^2 = (a_3^1)^2 + (a_3^2)^2 = 1-(a_3^3)^2, \tag{32}$$

or,

$$2(a_2^3a_3^1)^2 + (a_3^3)^2 - 1 = 0. \tag{33}$$

On the other hand, let θ, α denote angles between e_1 and E_2, between e_3 and E_1, respectively, since $a_1^1=0$, we have

$$\begin{cases}
e_1 = \cos\theta E_2 + \sin\theta E_3,\\
e_2 = \sin\alpha E_1 - \sin\theta\cos\alpha E_2 + \cos\theta\cos\alpha E_3,\\
e_3 = \cos\alpha E_1 + \sin\theta\sin\alpha E_2 - \cos\theta\sin\alpha E_3,
\end{cases} \tag{34}$$

where $a_1^2=\cos\theta,\ a_1^3=\sin\theta,\ a_2^1=\sin\alpha,\ a_2^2=-\sin\theta\cos\alpha,\ a_2^3=\cos\theta\cos\alpha,\ a_3^1=\cos\alpha,\ a_3^2=\sin\theta\sin\alpha$ and $a_3^3=-\cos\theta\sin\alpha$.
Using the 1st and the 3rd equation of (21), it is not difficult to check the following

$$e_1(\alpha) = -\sigma,\ e_1(\theta) = -\cos\theta. \tag{35}$$

Since $a_2^3=\cos\theta\cos\alpha,\ a_3^1=\cos\alpha$ and $a_3^3=-\cos\theta\sin\alpha$, then (33) turns into

$$\cos^2\theta(2\cos^4\alpha + \sin^2\alpha) - 1 = 0. \tag{36}$$

Note that θ is angle between e_1 and E_2, but α angle between e_3 and E_1, then the two functions: $\cos^2\theta,\ 2\cos^4\alpha+\sin^2\alpha$ are linearly independent. Then, Eq. (36) implies taht θ and α have to be constants, and hence $\cos\theta=0$ since (35). Substituting this into Eq. (36). we have $-1=0$, a contradiction.
Summarizing all results in the above cases, the theorem follows.

□

Remark 1. Let $\pi:(\mathbb{R}^3, g_{Sol}=e^{2z}\mathrm{d}x^2+e^{-2z}\mathrm{d}y^2+\mathrm{d}z^2)\to(N^2,h)$ be a Riemannian submersion with e_3 being vertical. If $e_3\neq\pm E_1$, i.e., $a_3^1\neq\pm1$, by Theorem 3.3, one can choose such an adapted frame $\{e_1=a_1^2E_2+a_1^3E_3,\ e_2,\ e_3\}$ to π and $f_2\neq0$. From these, the case corresponding to $a_1^1=f_1=f_3=0$, $f_2\neq0$, $a_3^1\neq\pm1$ and $a_2^1\neq0$. Clearly, this implies that the case $a_3^1\neq\pm1$, and $a_1^1=f_1=f_2=0$ can not happen.

Theorem 3.4. *A Riemannian submersion* $\pi:(\mathbb{R}^3, g_{Sol}=e^{2z}\mathrm{d}x^2+e^{-2z}\mathrm{d}y^2+\mathrm{d}z^2)\to(N^2,h)$ *from Sol space exists only in* $(\mathbb{R}^3,g_{Sol})\to H^2$ *with Gauss curvature of the base space* $K^N=-1$.

Proof. Let $\pi:(\mathbb{R}^3, g_{Sol}=e^{2z}\mathrm{d}x^2+e^{-2z}\mathrm{d}y^2+\mathrm{d}z^2)\to(N^2,h)$ be a Riemannian submersion with e_3 being vertical. For the above notations and signs, we just need to consider the two cases $a_3^1=0$ or $a_3^3=\pm1$. We use the proof by contradiction to obtain the theorem. We now assume $a_3^1\neq0$ and $a_3^3\neq\pm1$. It follows from Theorem 3.3 and Remark 1 that there exists such an adapted frame $\{e_1=a_1^2E_2+a_1^3E_3,\ e_2,\ e_3\}$ to π, and hence the following hold

$$a_1^1=f_1=f_3=0,\ f_2\neq0,\ a_2^1\neq0,\pm1 \text{ and } a_3^1\neq0,\pm1. \tag{37}$$

Then, (10) becomes as

$$\begin{cases}-e_1(\sigma)+2\kappa_1\sigma=-2a_2^3a_3^3,\\ e_1(\kappa_1)+\sigma^2-\kappa_1^2=2(a_2^3)^2-1,\\ e_1(\kappa_2)-e_3(\sigma)-\kappa_1\kappa_2=-2a_1^3a_2^3,\\ e_1(f_2)-f_2^2-3\sigma^2=2(a_3^3)^2-1,\\ -e_2(\sigma)+2\kappa_2\sigma=2a_1^3a_3^3,\\ e_2(\kappa_1)+e_3(\sigma)+\kappa_2f_2-\kappa_1\kappa_2=-2a_1^3a_2^3,\\ \sigma^2+e_2(\kappa_2)-\kappa_1f_2-\kappa_2^2=2(a_1^3)^2-1.\end{cases} \tag{38}$$

We apply e_1 to both sides the 12th equation of (21), together with $f_3=0$, to get

$$e_1(\kappa_1)a_3^1+\kappa_1e_1(a_3^1)-e_1(\sigma)a_2^1-\sigma e_1(a_2^1)-e_1(a_2^2)=0. \tag{39}$$

Using the 3rd, the 4th, the 6th equation of (21), the 1st, the 2nd equation of (38) and the 1st equation of (9), a straightforward computation gives

(40)
$$\begin{aligned}
0 &= (\kappa_1^2-\sigma^2+2(a_2^3)^2-1)a_3^1+\kappa_1\sigma a_2^1-(2\kappa_1\sigma+2a_2^3a_3^3)a_2^1+\sigma^2a_3^1-a_1^2a_2^3+\sigma a_3^2\\
&= \kappa_1^2a_3^1-\sigma^2a_3^1+2a_3^1(a_2^3)^2-a_3^1+\kappa_1\sigma a_2^1-2\kappa_1\sigma a_2^1-2a_2^3a_3^3a_2^1+\sigma^2a_3^1-a_1^2a_2^3+\sigma a_3^2.
\end{aligned}$$

Substituting the 12th equation of (21) into (39), together with $a_2^1a_2^3+a_3^1a_3^3=0$, $f_3=0$ and $a_2^2=a_1^3a_3^1$, a direct computation gives

(41)
$$\begin{aligned}
0 &= \kappa_1(\sigma a_2^1+a_2^2)-\sigma^2a_3^1+2a_3^1(a_2^3)^2-a_3^1+\kappa_1\sigma a_2^1-2\kappa_1\sigma a_2^1+2a_3^1(a_3^3)^2+\sigma^2a_3^1-a_1^2a_2^3+\sigma^2a_3^1\\
&= \kappa_1a_2^2+2a_3^1[(a_2^3)^2+(a_3^3)^2]-a_3^1-a_1^2a_2^3+\sigma^2a_3^1\\
&= \kappa_1a_3^1a_1^3+2a_3^1[1-(a_1^3)^2]-a_3^1-a_1^2a_2^3+\sigma^2a_3^1\\
&= (\sigma a_2^1+a_2^2)a_1^3+2a_3^1-2a_3^1(a_1^3)^2-a_3^1-a_1^2a_2^3+\sigma^2a_3^1\\
&= \sigma a_2^1a_1^3+a_2^2a_1^3-2a_3^1(a_1^3)^2+a_3^1-a_1^2a_2^3+\sigma^2a_3^1\\
&= \sigma^2a_3^1+\sigma a_3^2-2a_3^1(a_1^3)^2,
\end{aligned}$$

the last equality holds for using the fact $a_2^1a_1^3=a_3^2$ and $a_3^1=a_1^2a_2^3-a_1^3a_2^2$.

Since $a_1^1=0$, one can assume that

$$\begin{cases}
e_1=\cos\theta E_2+\sin\theta E_3,\\
e_2=\sin\alpha E_1-\sin\theta\cos\alpha E_2+\cos\theta\cos\alpha E_3,\\
e_3=\cos\alpha E_1+\sin\theta\sin\alpha E_2-\cos\theta\sin\alpha E_3,
\end{cases}\tag{42}$$

where $a_1^2=\cos\theta,\ a_1^3=\sin\theta,\ a_2^1=\sin\alpha,\ a_2^2=-\sin\theta\cos\alpha,\ a_2^3=\cos\theta\cos\alpha,\ a_3^1=\cos\alpha,\ a_3^2=\sin\theta\sin\alpha$ and $a_3^3=-\cos\theta\sin\alpha$.

We applying the 1st and the 3rd equation of (21) to see that

$$e_1(\alpha)=-\sigma,\ e_1(\theta)=-\cos\theta.\tag{43}$$

Since $a_1^3=\sin\theta,\ a_3^1=\cos\alpha$ and $a_3^2=\sin\theta\sin\alpha$, then (41) becomes

$$\sigma^2\cos\alpha+\sigma\sin\theta\sin\alpha-2\cos\alpha\sin^2\theta=0.\tag{44}$$

One solves the above equation to obtain

$$\sigma=\sin\theta\varphi(\alpha),\tag{45}$$

where $\varphi(\alpha)=\frac{-\sin\alpha\pm\sqrt{1+7\cos^2\alpha}}{\cos\alpha}$.

Substituting $a_2^1=\sin\alpha,\ a_2^2=-\sin\theta\cos\alpha,\ a_3^1=\cos\alpha,\ f_3=0$ and (45) into the 12th equation of (21), we have

$$\kappa_1=\sin\theta\psi(\alpha),\tag{46}$$

where denote by $\psi(\alpha)=-\cos\alpha+\sin\alpha\varphi(\alpha)$ and $\varphi(\alpha)=\frac{-\sin\alpha\pm\sqrt{1+7\cos^2\alpha}}{\cos\alpha}$.

We substitute (45) and (46) into the 1st equation of (38) to have

$$e_1(\sigma)=2\kappa_1\sigma+2a_2^3a_3^3=2\sin^2\theta\varphi(\alpha)\psi(\alpha)-2\cos^2\theta\cos\alpha\sin\alpha, \tag{47}$$

where $\psi(\alpha)=-\cos\alpha+\sin\alpha\varphi(\alpha)(\alpha)$ and $\varphi(\alpha)=\frac{-\sin\alpha\pm\sqrt{1+7\cos^2\alpha}}{\cos\alpha}$.
On the other hand, substitute (45) into the left-hand side of (47) to compute as

$$\begin{aligned} e_1(\sigma)&=e_1\left(\sin\theta\varphi(\alpha)\right)=\cos\theta\varphi(\alpha)e_1(\theta)+\sin\theta\varphi'(\alpha)e_1(\alpha)\\ &=-\cos^2\theta\varphi(\alpha)-\sin^2\theta\varphi'(\alpha)\varphi(\alpha), \end{aligned} \tag{48}$$

the last holds by using (43).

Comparing (47) with (48), we deduce

$$\sin^2\theta\{2\varphi(\alpha)\psi(\alpha)+\varphi'(\alpha)\varphi(\alpha)\}+\cos^2\theta\{-2\cos\alpha\sin\alpha+\varphi(\alpha)\}=0. \tag{49}$$

Solving the above equation, we obtain a contradiction. Indeed, if θ is a constant, by the 2nd equation of (43), we have $\cos\theta=0$ and $\sin\theta=\pm1$. Substituting this into the above equation we have

$$2\varphi(\alpha)\psi(\alpha)+\varphi'(\alpha)\varphi(\alpha)=0, \tag{50}$$

this implies α being constant and hence $\sigma=0$ together with the 1st equation of (43). Hence, using (44), we get $a_3^1=\cos\alpha=0$ since $\sin\theta=\pm1$, a contradiction.

If $\theta\neq$ constant, but the two functions $\sin^2\theta,\ \cos^2\theta$ are linearly independent, then (49) means

$$2\varphi(\alpha)\psi(\alpha)+\varphi'(\alpha)\varphi(\alpha)=0,\ \text{and}\ -2\cos\alpha\sin\alpha+\varphi(\alpha)=0, \tag{51}$$

where $\psi(\alpha)=-\cos\alpha+\sin\alpha\varphi(\alpha)$ and $\varphi(\alpha)=\frac{-\sin\alpha\pm\sqrt{1+7\cos^2\alpha}}{\cos\alpha}$.
Clearly, the second equation of the above equation implies α being a constant and hence $\sigma=0$ together with the 1st equation of (43). Moreover, by (44) and $a_3^1=\cos\alpha\neq0$, one finds that $\sin\theta=0$ and hence θ is a constant contradicting the assumption $\theta\neq$ constant. From these, when $a_3^1\neq\pm1,0$, there is no a Riemannaian submersion $\pi:(\mathbb{R}^3,g_{Sol})\to(N^2,h)$ from Sol space no matter what (N^2,h) is.
In addition, if $a_3^1=0$, we have $a_2^1=\pm1$ since $a_1^1=0$, in this case, a straightforward computation similar to those used computing Case II in Theorem 3.5 gives $f_1=f_3=0$, $f_2=-1$ and hence Gauss curvature of the base space $K^N=e_1(f_2)-f_2^2=-1$; if $a_3^1=\pm1$, we have $a_2^1=0$ since $a_1^1=0$, in this case, a direct calculation similar to those used calculating Case I in Theorem 3.5 gives $f_1=0$, $f_2=1$ and hence Gauss curvature of the base space $K^N=e_1(f_2)-f_2^2=-1$. Clearly, This

implies that the a Riemannaian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ exists only in $(\mathbb{R}^3, g_{Sol}) \to H^2$ with Gauss curvature of the base space $K^N = -1$.
From which we obtain the theorem.

$\square$

Theorem 3.5. *There exists no biharmonic Riemannaian submersion* $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ *no matter what* (N^2, h) *is.*

Proof. Let ∇ denote the Levi-Civita connection on Sol space $(\mathbb{R}^3, g_{Sol})$ with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$ and e_3 being vertical. We denote by $e_i = \sum\limits_{j=1}^{3} a_i^j E_j,\ i = 1, 2, 3.$ To complete the proof of the theorem, from Theorem 3.4, we only discuss biharmonicity of a Riemannaian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to H^2$. Furthermore, from the proof of Theorem 3.4, we only need to consider the two cases $a_3^1 = \pm 1$ or $a_3^1 = 0$.
Case I: $a_3^1 = \pm 1$. In this case, one sees that $e_3 = \pm E_1$ and hence take an orthogonal frame $\{e_1 = E_3,\ e_2 = E_2,\ e_3 = -E_1\}$ on $(\mathbb{R}^3, g_{Sol})$ with e_3 being vertical. A direct computation using (2) and (3) gives

$$\begin{aligned}
&[e_1, e_2] = e_2,\ [e_1, e_3] = -e_3,\ [e_2, e_3] = 0,\\
&\nabla_{e_2} e_1 = -e_2,\ \ \nabla_{e_2} e_2 = e_1,\ \nabla_{e_3} e_1 = e_3,\ \nabla_{e_3} e_3 = -e_1,\\
&\text{all other } \nabla_{e_i} e_j = 0,\ i, j = 1, 2, 3.
\end{aligned}$$

It follows that the (generalized) integrability data $f_1 = f_3 = \kappa_2 = \sigma = 0, \kappa_1 = -f_2 = -1$ and hence $\{e_1 = E_3,\ e_2 = E_2,\ e_3 = E_1\}$ is actually adapted to π with e_3 being vertical. Then, biharmonic equation (15) reduces to

$$\Delta\kappa_1 - \kappa_1\{-K^N + f_2^2\} = 0. \tag{52}$$

However, the left-hand term of (52) can be computed as

$$\begin{aligned}
\Delta\kappa_1 - \kappa_1\{-K^N + f_2^2\} &= \sum_{i=1}^{3} \left(e_i e_i(\kappa_1) - \nabla_{e_i} e_i(\kappa_1)\right) - \kappa_1\{-e_1(f_2) + 2f_2^2\}\\
&= 0 + 1 \times 2 = 2 \neq 0.
\end{aligned} \tag{53}$$

Therefore, the Riemannian submersion π is not biharmonic in this case.

Case II: $a_3^1 = 0$. In this case, we have $a_2^1 = \pm 1$ since $a_1^1 = 0$. Then, we can take an orthonormal frame $\{e_1 = E_3,\ e_2 = E_1,,\ e_3 = E_2\}$ with e_3 being vertical. A direct computation using (2) and (3) gives

$$\begin{aligned}
&[e_1, e_2] = -e_2,\ [e_1, e_3] = e_3,\ [e_2, e_3] = 0,\\
&\nabla_{e_2} e_1 = -e_2,\ \nabla_{e_2} e_2 = -e_1,\ \nabla_{e_3} e_1 = -e_3,\ \nabla_{e_3} e_3 = e_1,\\
&\text{all other } \nabla_{e_i} e_j = 0,\ i, j = 1, 2, 3.
\end{aligned} \tag{54}$$

This follows that the (generalized) integrability data $f_1 = f_3 = \kappa_2 = \sigma = 0, \kappa_1 = -f_2 = 1$ and hence $\{e_1 = E_3,\ e_2 = E_1, ,\ e_3 = E_2\}$ becomes adapted to π with e_3 being vertical. Substituting this into biharmonic equation (15) and a direct computation, we have

$$\begin{aligned} 0 &= \Delta\kappa_1 - \kappa_1\{-K^N + f_2^2\} \\ &= \sum_{i=1}^{3}(e_ie_i(\kappa_1) - \nabla_{e_i}e_i(\kappa_1)) - \kappa_1\{-e_1(f_2) + 2f_2^2\} = 0 + 1 \times 2 = 2, \end{aligned}$$

which is a contradiction. Thus, the Riemannian submersion π is not biharmonic. Summarizing all results in the above cases we obtain the theorem. □

Remark 2. We would like to point out that, with respect to local coordinates, a Riemannaian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to H^2$ can be locally expressed as the following (up to equivalence):
(a): the Riemannaian submersion

$$\pi : (\mathbb{R}^3, g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2) \to (\mathbb{R}^2, e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2),\ \pi(x, y, z) = (y, z),$$

or,
(b): the Riemannaian submersion

$$\pi : (\mathbb{R}^3, g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2) \to (\mathbb{R}^2, e^{2z}\mathrm{d}x^2 + \mathrm{d}z^2),\ \pi(x, y, z) = (x, z).$$

By Theorem 2.2 and Theorem 3.5, these Riemannaian submersions are neither harmonic nor biharmonic.

As a consequence of Theorem 2.2 and Theorem 3.5, we state the following fact

Corollary 3.6. *Any Riemannaian submersion $\pi : (\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ from Sol space to a surface is neither harmonic nor biharmonic.*

Although there is no (harmonic) biharmonic Riemannaian submersion from Sol space to a surface, there exist many (harmonic) biharmonic maps $(\mathbb{R}^3, g_{Sol}) \to (N^2, h)$ which are not Riemannaian submersions.

Example 1. The maps $\phi : (\mathbb{R}^3, g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2) \to (\mathbb{R}^2, du^2 + dv^2)$, $\phi(x, y, z) = (u, v) = (y, Az^3 + Bz^2 + Cz + D)$ are biharmonic, where A, B, C, D are constants. In particular, when $A^2 + B^2 > 0$, this family of maps are proper biharmonic. Note that these maps are not Riemannaian submersions.

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